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On Starlikeness and Convexity of Certain Analytic Functions

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Abstract. Two subclasses $S(\alpha, b)$ and $C(\alpha, b)$ of the class A consisting of all analytic functions with $f(0)=0$ and $f'(0)=1$ in the open unit disk U are introduced. The classes $S(\alpha, b)$ and $C(\alpha, b)$ are the generalization classes of classes defined by H.Silverman (cf.[2]) and by T.Sekine and S.Owa (cf.[1]). The object of the present paper is to derive some coefficient inequalities for functions belonging to $S(\alpha, b)$ and $C(\alpha, b)$. Also we consider some necessary conditions for $f(z)$ belonging to the classes $S(\alpha, b)$ and $C(\alpha, b)$. Some interesting examples for our results are also given.

I. Introduction.

Let A denote the class of functions $f(z)$ of the form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

which are analitic in the open unit disk $U=\{z : z \in \mathbb{C} \text{ and } |z|<1\}$.

A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha$$

for some α $(0 \leqq \alpha < 1)$ and for all z in U. We denote by $S^*(\alpha)$ the subclass of A consisting of all starlike functions $f(z)$ of order α in U.

Further, a function $f(z) \in A$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in U$. We denote by $K(\alpha)$ the subclass of A consisting of such convex functions $f(z)$ of order α in U.

Let $S(\alpha, b)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$\left|\frac{zf'(z)}{f(z)}-b\right|<\mathrm{Re}(b)-\alpha \quad (0 \leqq \alpha < \mathrm{Re}(b)-|b-1|)$$

for $b \in \mathbb{C}$.

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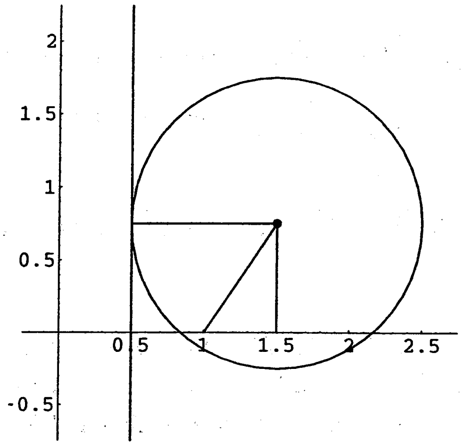

Figure 1: Image of $\left|\dfrac{zf'(z)}{f(z)} - b\right| < \mathrm{Re}(b) - \alpha.$

Let $C(\alpha, b)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$\left|1 + \frac{zf''(z)}{f'(z)} - b\right| < \mathrm{Re}(b) - \alpha \quad (0 \leqq \alpha < \mathrm{Re}(b) - |b-1|)$$

for $b \in \mathbb{C}$.

Remark 1. Letting $b = b_1 + ib_2$, the condition

$$0 \leqq \alpha < \mathrm{Re}(b) - |b-1|$$

shows that

$$0 \leqq b_2^2 < (1-\alpha)(2b_1 - (1+\alpha)).$$

Therefore, b should be in the right half plane of the parabola.

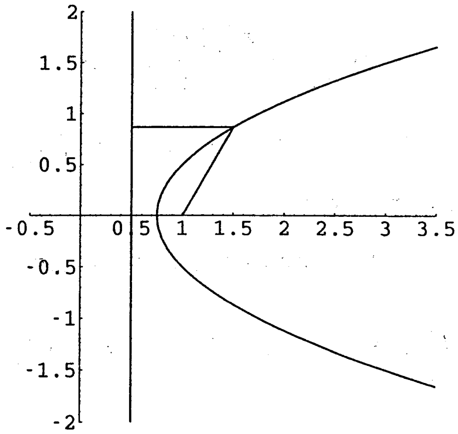

Figure 2: Range of b, that is, $|b-1| < \mathrm{Re}(b) - \alpha$.

Remark 2. (1) In 1975, Silverman [2] has showed that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n| \leqq 1-\alpha,$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha \quad (z \in U),$$

so $f(z) \in S(\alpha,1)$.

(2) Sekine and Owa [1] have showed that

(i) if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n| \leqq 1-\alpha$$

for $1 \leqq b \leqq 2$, then

$$\left|\frac{zf'(z)}{f(z)}-b\right| < b-\alpha \quad (z \in U),$$

so $f(z) \in S(\alpha,b)$,

(ii) if $f(z) \in A$ satisfies

$$\sum_{n=2}^{j}(2b-n-\alpha)\,|a_n| + \sum_{n=j+1}^{\infty}(n-\alpha)\,|a_n| \leqq 1-\alpha$$

for $b > 2$, then

$$\left|\frac{zf'(z)}{f(z)}-b\right| < b-\alpha \quad (z \in U),$$

so $f(z) \in S(\alpha,b)$,

(iii) if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n| \leqq 2b-1-\alpha$$

for $\dfrac{1+\alpha}{2} < b < 1$, then

$$\left|\frac{zf'(z)}{f(z)}-b\right| < b-\alpha \quad (z \in U),$$

so $f(z) \in S(\alpha,b)$.

II. Coefficient Inequalities.

We shall now prove the following theorems in a same way of Theorem 1 of Silverman [2], or Sekine and Owa [1].

Theorem 1. *Let $f(z) \in A$, $b \in \mathbb{C}$ and $0 \leqq \alpha < \mathrm{Re}(b) - |b-1|$. If $f(z)$ satisfies*

$$\sum_{n=2}^{\infty} \{|n-b| + \mathrm{Re}(b) - \alpha\} |a_n| \leqq \mathrm{Re}(b) - \alpha - |b-1|,$$

then

$$\left| \frac{zf'(z)}{f(z)} - b \right| < \mathrm{Re}(b) - \alpha,$$

that is, $f(z) \in S(\alpha, b)$.

Proof. For $f(z) \in A$, it follows that

$$\begin{aligned}
\left| \frac{zf'(z)}{f(z)} - b \right| &= \left| \frac{1 - b + \sum_{n=2}^{\infty}(n-b)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\
&= \left| \frac{b - 1 - \sum_{n=2}^{\infty}(n-b)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\
&\leqq \frac{|b-1| + \sum_{n=2}^{\infty} |n-b||a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n||z|^{n-1}} \\
&< \frac{|b-1| + \sum_{n=2}^{\infty} |n-b||a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}.
\end{aligned}$$

The last expression is bounded by $\mathrm{Re}(b) - \alpha$ if

$$|b-1| + \sum_{n=2}^{\infty} |n-b||a_n| \leqq (\mathrm{Re}(b) - \alpha) \left(1 - \sum_{n=2}^{\infty} |a_n| \right),$$

which is equivalent to

$$\sum_{n=2}^{\infty} \{|n-b| + \mathrm{Re}(b) - \alpha\} |a_n| \leqq \mathrm{Re}(b) - \alpha - |b-1|.$$

Hence we have $\left| \frac{zf'(z)}{f(z)} - b \right| < \mathrm{Re}(b) - \alpha$, and the Theorem 1 is proved. □

Remark 3. If we take $b = 1$ in Theorem 1, then we have Theorem 1 by Silverman [2]. Further, if we take some real b such that $b > \frac{1+\alpha}{2}$, then we have Theorem 1 by Sekine and Owa [1].

Corollary 1. *Let $f(z) \in A$, $b \in \mathbb{C}$ and $0 \leqq \alpha < \mathrm{Re}(b) - |b-1|$. If $f(z)$ satisfies*

$$\sum_{n=2}^{\infty} n\{|n-b| + \mathrm{Re}(b) - \alpha\}\,|a_n| \leqq \mathrm{Re}(b) - \alpha - |b-1|,$$

then

$$\left|1 + \frac{zf''(z)}{f'(z)} - b\right| < \mathrm{Re}(b) - \alpha,$$

that is, $f(z) \in C(\alpha, b)$.

Proof. It is well known that $f(z) \in C(\alpha, b)$ if and only if $zf'(z) \in S(\alpha, b)$. Because, suppose that $F(z) = zf'(z)$. Then

$$\begin{aligned}\left|\frac{zF'(z)}{F(z)} - b\right| &= \left|\frac{z(f'(z) + zf''(z))}{zf'(z)} - b\right| \\ &= \left|\frac{f'(z) + zf''(z)}{f'(z)} - b\right| \\ &= \left|1 + \frac{zf''(z)}{f'(z)} - b\right| < \mathrm{Re}(b) - \alpha.\end{aligned}$$

Since $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$, we may replace a_n with na_n in Theorem 1.
This completes the proof of Corollary 1. □

Remark 4. If we take $b = 1$ in Corollary 1, then we have the result by Silverman [2]. And if we take some real b such that $b > \dfrac{1+\alpha}{2}$, then we have the result by Sekine and Owa [1].

III. Distortion Inequalities.

If we consider the function $f(z) \in S\left(\frac{1}{4}, \frac{3}{4}\right)$, then

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{4}\right| < \frac{1}{2} \quad (z \in U).$$

Recently, Silverman [3] has given the function $f(z) = z - \frac{1}{3}z^2$ which is in the class $S\left(\frac{1}{4}, \frac{3}{4}\right)$. But this function $f(z)$ dose not satisfy the coefficient inequality of Theorem 1. As we mention the above, the inverse of Theorem 1 is not true in general.

Now, let $S_0(\alpha, b)$ be the subclass of $S(\alpha, b)$ consisting of $f(z)$ satisfying the coefficient inequality of Theorem 1. Further, let $C_0(\alpha, b)$ be the subclass of $C(\alpha, b)$ consisting of $f(z)$ which satisfy the coefficient inequality of Corollary 1.

Theorem 2. *If $f(z) \in S_0(\alpha, b)$, then*

$$|z| - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}|z|^2 \leqq |f(z)| \leqq |z| + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}|z|^2 \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}\, z^2 \qquad (z = \pm|z|).$$

Proof. By the assumption $f(z) \in S_0(\alpha, b)$, we note that

$$\left\{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\right\} \sum_{n=2}^{\infty} |a_n| \leqq \sum_{n=2}^{\infty} \{|n-b| + \mathrm{Re}(b) - \alpha\}|a_n| \leqq \mathrm{Re}(b) - \alpha - |b-1|,$$

that is,

$$\sum_{n=2}^{\infty} |a_n| \leqq \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}.$$

Thus, using the preceding result, we have

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leqq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \\ &\leqq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\leqq |z| + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}|z|^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \\ &\geqq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\ &\geqq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\geqq |z| - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}|z|^2. \end{aligned}$$

Consequently, if $f(z) \in S_0(\alpha, b)$, then we obtain

$$|z| - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z|^2 \leqq |f(z)| \leqq |z| + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z|^2.$$

Finally, taking the function

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} z^2,$$

we have the equalities for $z = |z|$, and for $z = -|z|$, respectively.

This completes the proof of Theorem 2. □

For $f'(z)$ of $f(z)$ belonging to $S_0(\alpha, b)$, we have

Theorem 3. *If $f(z) \in S_0(\alpha, b)$, then*

$$1 - \frac{2(\mathrm{Re}(b) - \alpha - |b-1|)}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z| \leqq |f'(z)| \leqq 1 + \frac{2(\mathrm{Re}(b) - \alpha - |b-1|)}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z| \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} z^2 \qquad (z = \pm|z|).$$

Proof. By the assumption $f(z) \in S_0(\alpha, b)$, we note that

$$\begin{aligned}
\sum_{n=2}^{\infty} \{|n-b| + \mathrm{Re}(b) - \alpha\} |a_n| &= \sum_{n=2}^{\infty} \left\{ n \left| 1 - \frac{b}{n} \right| + \mathrm{Re}(b) - \alpha \right\} |a_n| \\
&= \sum_{n=2}^{\infty} \left\{ n \left| 1 - \frac{b}{n} \right| \right\} |a_n| + \sum_{n=2}^{\infty} (\mathrm{Re}(b) - \alpha) |a_n| \\
&\leqq \mathrm{Re}(b) - \alpha - |b-1|,
\end{aligned}$$

which implies that

$$\min_{n \geqq 2} \left| 1 - \frac{b}{n} \right| \sum_{n=2}^{\infty} n|a_n| + \sum_{n=2}^{\infty} (\mathrm{Re}(b) - \alpha)|a_n| \leqq \sum_{n=2}^{\infty} \left\{ n \left| 1 - \frac{b}{n} \right| \right\} |a_n| + \sum_{n=2}^{\infty} (\mathrm{Re}(b) - \alpha)|a_n|.$$

This gives us that

$$\begin{aligned}
\min_{n \geqq 2} \left| 1 - \frac{b}{n} \right| \sum_{n=2}^{\infty} n|a_n| &\leqq \mathrm{Re}(b) - \alpha - |b-1| - (\mathrm{Re}(b) - \alpha) \sum_{n=2}^{\infty} |a_n| \\
&\leqq \mathrm{Re}(b) - \alpha - |b-1| - (\mathrm{Re}(b) - \alpha) \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} \\
&\leqq \frac{(\mathrm{Re}(b) - \alpha - |b-1|) \min_{n \geqq 2} |n-b|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha},
\end{aligned}$$

that is, that

$$\sum_{n=2}^{\infty} n|a_n| \leqq \frac{(\mathrm{Re}(b)-\alpha-|b-1|)\min_{n\geqq 2}|n-b|}{\min_{n\geqq 2}|n-b|+\mathrm{Re}(b)-\alpha} \times \frac{1}{\min_{n\geqq 2}\left|1-\dfrac{b}{n}\right|}$$

$$= \frac{2(\mathrm{Re}(b)-\alpha-|b-1|)}{\min_{n\geqq 2}|n-b|+\mathrm{Re}(b)-\alpha}.$$

Because, $\min_{n\geqq 2}|n-b|$ has the minimum value when $\mathrm{Re}(b)$ is the nearest n, and $\min_{n\geqq}\left|1-\dfrac{b}{n}\right|$ has the minimum value when $\mathrm{Re}\left(\dfrac{b}{n}\right)$ is the nearest 1, in other words, when $\mathrm{Re}(b)$ is the nearest n. Consequently, we have $\dfrac{\min_{n\geqq 2}|n-b|}{\min_{n\geqq 2}\left|1-\dfrac{b}{n}\right|} = 2.$

In view the above, we see that

$$|f'(z)| = \left|1+\sum_{n=2}^{\infty} n\ a_n z^{n-1}\right|$$

$$\leqq 1+\sum_{n=2}^{\infty} n|a_n||z|^{n-1}$$

$$\leqq 1+|z|\sum_{n=2}^{\infty} n|a_n|$$

$$\leqq 1+\frac{2(\mathrm{Re}(b)-\alpha-|b-1|)}{\min_{n\geqq 2}|n-b|+\mathrm{Re}(b)-\alpha}|z|.$$

Similarly, we have

$$|f'(z)| = \left|1+\sum_{n=2}^{\infty} n\ a_n z^{n-1}\right|$$

$$\geqq 1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1}$$

$$\geqq 1-|z|\sum_{n=2}^{\infty} n|a_n|$$

$$\geqq 1-\frac{2(\mathrm{Re}(b)-\alpha-|b-1|)}{\min_{n\geqq 2}|n-b|+\mathrm{Re}(b)-\alpha}|z|.$$

Consequently, if $f(z) \in S_0(\alpha, b)$, then we obtain

$$1 - \frac{2(\mathrm{Re}(b) - \alpha - |b-1|)}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}|z| \leqq |f'(z)| \leqq 1 + \frac{2(\mathrm{Re}(b) - \alpha - |b-1|)}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}|z|.$$

Taking $f(z)$ given by

$$f'(z) = 1 + \frac{2(\mathrm{Re}(b) - \alpha - |b-1|)}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}\, z,$$

which is equivalent to

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}\, z^2,$$

we see that the equalities in Theorem 3 are attained. □

Now, we give an example for Theorem 3.

Example 1. *Let $f(z) \in S_0(\alpha, b), b = 3 + \frac{1}{2} i$ and $0 \leqq \alpha = \frac{4}{5} < 3 - \frac{\sqrt{17}}{2}$. Then we have*

$$f(z) = z + \frac{22 - 5\sqrt{17}}{27}\, z^2 \text{ and } f'(z) = 1 + \frac{2(22 - 5\sqrt{17})}{27}\, z.$$

Therefore

$$\frac{z\, f'(z)}{f(z)} = 2 - \frac{27}{27 + (22 - 5\sqrt{17})\, z}.$$

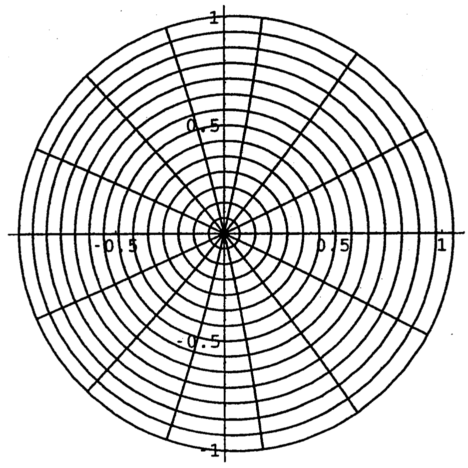

Figure 3: Image of U by $f(z)$ for Example 1.

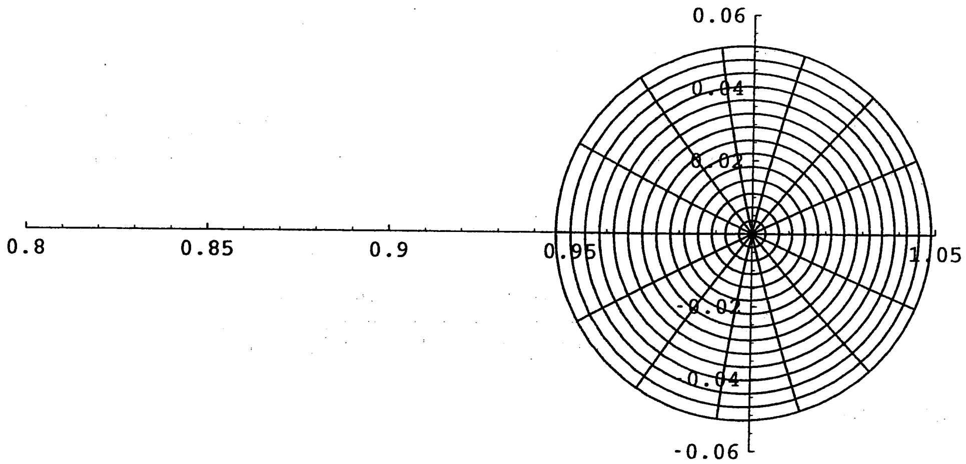

Figure 4: Image of U by $\dfrac{zf'(z)}{f(z)}$ for Example 1.

Furthermove, for $f(z)$ in the class $C_0(\alpha, b)$, we see

Theorem 4. *If* $f(z) \in C_0(\alpha\ , b)$, *then*

$$|z| - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2}|n-b| + \mathrm{Re}(b) - \alpha\}}|z|^2 \leqq |f(z)| \leqq |z| + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2}|n-b| + \mathrm{Re}(b) - \alpha\}}|z|^2 \quad (z \in U).$$

Equality holds for the function $f(z)$ *given by*

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2}|n-b| + \mathrm{Re}(b) - \alpha\}}\ z^2 \qquad (z = \pm|z|).$$

Proof. For $f(z) \in C_0(\alpha, b)$, we note that

$$2\left\{\min_{n\geqq 2}|n-b| + \mathrm{Re}(b) - \alpha\right\}\sum_{n=2}^{\infty}|a_n| \leqq \sum_{n=2}^{\infty} n\{|n-b| + \mathrm{Re}(b) - \alpha\}|a_n| \leqq \mathrm{Re}(b) - \alpha - |b-1|,$$

that is, that

$$\sum_{n=2}^{\infty}|a_n| \leqq \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2}|n-b| + \mathrm{Re}(b) - \alpha\}}.$$

Applying the above inequality, we have

$$|f(z)| = \left|z + \sum_{n=2}^{\infty} a_n z^n\right|$$

$$\leqq |z| + \sum_{n=2}^{\infty}|a_n||z|^n$$

$$\leqq |z| + |z|^2\sum_{n=2}^{\infty}|a_n|$$

$$\leqq |z| + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2}|n-b| + \mathrm{Re}(b) - \alpha\}}|z|^2.$$

Similarly, we have

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right|$$

$$\geqq |z| - \sum_{n=2}^{\infty} |a_n||z|^n$$

$$\geqq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\geqq |z| - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\}} |z|^2.$$

Consequently, if $f(z) \in C_0(\alpha, b)$, then we obtain

$$|z| - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\}} |z|^2 \leqq |f(z)| \leqq |z| + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\}} |z|^2.$$

Finally, letting

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\}} z^2,$$

we know that the equalities of the theorem are attainded. □

For the derivative $f'(z)$ of $f(z)$ in $C_0(\alpha, b)$, we also have

Theorem 5. *If $f(z) \in C_0(\alpha, b)$, then*

$$1 - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z| \leqq |f'(z)| \leqq 1 + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z| \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\}} z^2 \quad (z = \pm|z|).$$

Proof. Noting that

$$\left\{ \min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha \right\} \sum_{n=2}^{\infty} n|a_n| \leqq \sum_{n=2}^{\infty} n\{|n-b| + \mathrm{Re}(b) - \alpha\}|a_n| \leqq \mathrm{Re}(b) - \alpha - |b-1|,$$

for $f(z) \in C_0(\alpha, b)$, we have

$$\sum_{n=2}^{\infty} n|a_n| \leqq \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n\geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}.$$

Thus, we have

$$|f'(z)| = \left|1 + \sum_{n=2}^{\infty} n\ a_n z^{n-1}\right|$$

$$\leqq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1}$$

$$\leqq 1 + |z| \sum_{n=2}^{\infty} n|a_n|$$

$$\leqq 1 + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z|,$$

and

$$|f'(z)| = \left|1 + \sum_{n=2}^{\infty} n\ a_n z^{n-1}\right|$$

$$\geqq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}$$

$$\geqq 1 - |z| \sum_{n=2}^{\infty} n|a_n|$$

$$\geqq 1 - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z|.$$

Consequently, if $f(z) \in C_0(\alpha, b)$, then we obtain

$$1 - \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z| \leqq |f'(z)| \leqq 1 + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha} |z|.$$

Making

$$f'(z) = 1 + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha}\ z,$$

or

$$f(z) = z + \frac{\mathrm{Re}(b) - \alpha - |b-1|}{2\{\min_{n \geqq 2} |n-b| + \mathrm{Re}(b) - \alpha\}}\ z^2,$$

we complete the proof of Theorem 5. □

For Theorem 5, we give the following example.

Example 2. *Let* $f(z) \in C_0(\alpha\ , b)$, $b = 3 + \frac{1}{2}i$ *and* $0 \leqq \alpha = \frac{4}{5} < 3 - \frac{\sqrt{17}}{2}$. *Then we have*

$$f(z) = z + \frac{22 - 5\sqrt{17}}{54}\, z^2,\ f'(z) = 1 + \frac{22 - 5\sqrt{17}}{27}\, z \text{ and } f''(z) = \frac{22 - 5\sqrt{17}}{27}.$$

Therefore

$$1 + \frac{z\ f''(z)}{f'(z)} = 2 - \frac{27}{27 + (22 - 5\sqrt{17})\ z}.$$

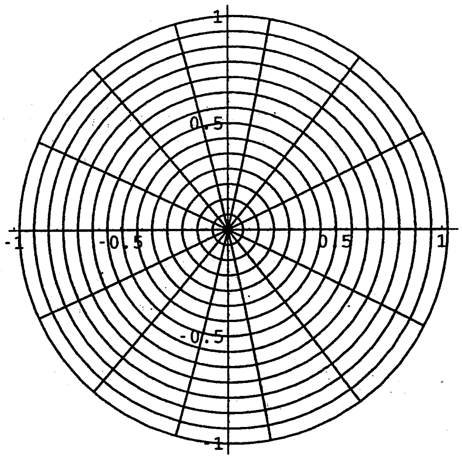

Figure 5: Image of U by $f(z)$ for Example 2.

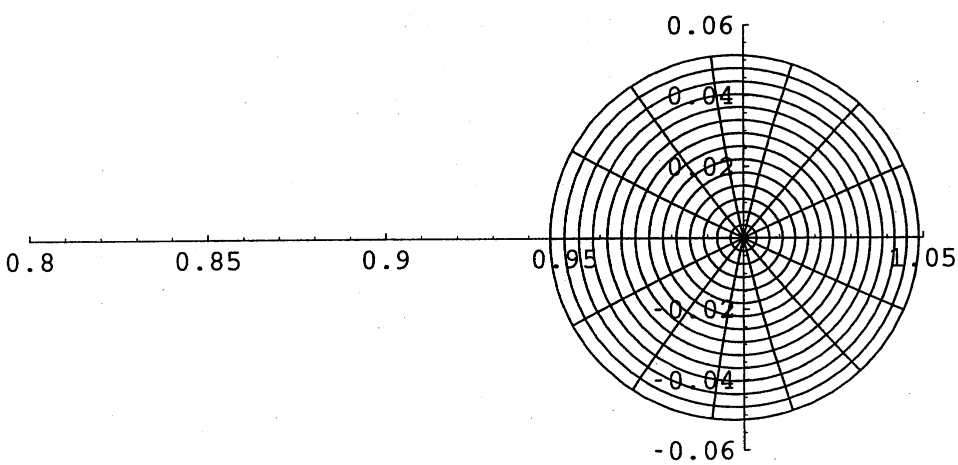

Figure 6: Image of U by $1 + \dfrac{zf''(z)}{f'(z)}$ for Example 2.

IV. Necessary conditions for the class $S(\alpha, b)$.

In general, we know that the coefficient inequalities which we give in Theorem 1 and Corollary 1 are not necessary conditions for the classes $S(\alpha, b)$ and $C(\alpha, b)$. Therefore, we try to find some necessary conditions for the class $S(\alpha, b)$.

Theorem 6. *Let $f(z)$ be in the class $S(\alpha, b)$ with $a_n = |a_n|e^{in\pi}$ and $b = 1 + ib_2$, then*

$$\left(\sum_{n=2}^{\infty}(n+\alpha)|a_n| + \frac{(1-b_2-\alpha)(1+b_2-\alpha)}{\alpha-1}\right)$$

$$\times\left(\sum_{n=2}^{\infty}(n-2+\alpha)|a_n| - \frac{(1-b_2-\alpha)(1+b_2-\alpha)}{\alpha-1}\right) + b_2^2\left(\sum_{n=2}^{\infty}|a_n|\right)^2$$

$$< \frac{b_2^2(1-b_2-\alpha)(1+b_2-\alpha)}{(\alpha-1)^2}.$$

Proof. Since $f(z) \in S(\alpha, b)$ if and only if $\left|\frac{zf'(z)}{f(z)} - b\right| < \mathrm{Re}(b) - \alpha = 1 - \alpha$, we have, for $z = e^{i\pi}$,

$$\left|\frac{zf'(z)}{f(z)} - b\right| = \left|\frac{b-1-\sum_{n=2}^{\infty}(n-b)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}\right|$$

$$= \frac{|b-1+\sum_{n=2}^{\infty}(n-b)|a_n||}{1-\sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|b-1+\sum_{n=2}^{\infty}n|a_n| - \sum_{n=2}^{\infty}b|a_n||}{1-\sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|1+ib_2-1+\sum_{n=2}^{\infty}n|a_n| - \sum_{n=2}^{\infty}(1+ib_2)|a_n||}{1-\sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|ib_2+\sum_{n=2}^{\infty}n|a_n| - \sum_{n=2}^{\infty}|a_n| - ib_2\sum_{n=2}^{\infty}|a_n||}{1-\sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|\sum_{n=2}^{\infty}(n-1)|a_n| + ib_2(1-\sum_{n=2}^{\infty}|a_n|)|}{1-\sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{\sqrt{(\sum_{n=2}^{\infty}(n-1)|a_n|)^2 + b_2^2(1-\sum_{n=2}^{\infty}|a_n|)^2}}{1-\sum_{n=2}^{\infty}|a_n|} < 1-\alpha.$$

It follows from the above that

$$\sqrt{\left(\sum_{n=2}^{\infty}(n-1)|a_n|\right)^2 + b_2^2\left(1-\sum_{n=2}^{\infty}|a_n|\right)^2} < 1-\alpha-\sum_{n=2}^{\infty}(1-\alpha)|a_n|,$$

that is, that

$$\left(\sum_{n=2}^{\infty}(n-1)|a_n|\right)^2+b_2^2\left(1-\sum_{n=2}^{\infty}|a_n|\right)^2$$

$$<(1-\alpha)^2-2(1-\alpha)^2\sum_{n=2}^{\infty}|a_n|+\left(\sum_{n=2}^{\infty}(1-\alpha)|a_n|\right)^2.$$

It is easy to see that

$$\left(\sum_{n=2}^{\infty}(n+\alpha)|a_n|\right)\left(\sum_{n=2}^{\infty}(n-2+\alpha)|a_n|\right)+b_2^2\left(\sum_{n=2}^{\infty}|a_n|\right)^2$$

$$+(2-4\alpha+2\alpha^2-2b_2^2)\sum_{n=2}^{\infty}|a_n|$$

$$<(1-b_2-\alpha)(1+b_2-\alpha).$$

Thus we obtain

$$\left(\sum_{n=2}^{\infty}(n+\alpha)|a_n|+\frac{(1-b_2-\alpha)(1+b_2-\alpha)}{\alpha-1}\right)$$

$$\times\left(\sum_{n=2}^{\infty}(n-2+\alpha)|a_n|-\frac{(1-b_2-\alpha)(1+b_2-\alpha)}{\alpha-1}\right)+b_2^2\left(\sum_{n=2}^{\infty}|a_n|\right)^2$$

$$<\frac{b_2^2(1-b_2-\alpha)(1+b_2-\alpha)}{(\alpha-1)^2},$$

which derives the proof of the theorem. □

Noting that $\dfrac{1+\alpha}{2}<\mathrm{Re}(b)$, we consider the case of $b=b_1+ib_2$ with $b_1>\dfrac{1+\alpha}{2}$.

Theorem 7. *If $f(z)\in S(\alpha,b)$ with $a_n=|a_n|e^{in\pi}$ and $b=b_1+ib_2$, then*

$$\left(\sum_{n=2}^{\infty}(n-\alpha)|a_n|+\frac{(\alpha-b_1)^2-|b|^2+b_1}{\alpha-b_1}\right)\left(\sum_{n=2}^{\infty}(n-2b_1+\alpha)|a_n|+\frac{(\alpha-b_1)^2-|b|^2+b_1}{\alpha-b_1}\right)$$

$$+b_2^2\left(\sum_{n=2}^{\infty}|a_n|\right)^2+2(b_1-1)\sum_{n=2}^{\infty}n|a_n|$$

$$<|b|^2-1-\frac{|b|^2(|b|^2-2b_1)-b_1^2}{(\alpha-b_1)^2}.$$

Proof. Note that $f(z) \in S(\alpha, b)$ if and only if $\left|\frac{zf'(z)}{f(z)} - b\right| < \mathrm{Re}(b) - \alpha = b_1 - \alpha$. Letting $z = e^{i\pi}$, we have

$$\left|\frac{zf'(z)}{f(z)} - b\right| = \left|\frac{b - 1 - \sum_{n=2}^{\infty}(n-b)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}\right|$$

$$= \frac{|b - 1 + \sum_{n=2}^{\infty}(n-b)|a_n||}{1 - \sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|b - 1 + \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} b|a_n||}{1 - \sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|b_1 + ib_2 - 1 + \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty}(b_1 + ib_2)|a_n||}{1 - \sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|(b_1 - 1) + ib_2 + \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} b_1|a_n| - ib_2\sum_{n=2}^{\infty}|a_n||}{1 - \sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{|(b_1 - 1) + \sum_{n=2}^{\infty}(n - b_1)|a_n| + ib_2(1 - \sum_{n=2}^{\infty}|a_n|)|}{1 - \sum_{n=2}^{\infty}|a_n|}$$

$$= \frac{\sqrt{(b_1 - 1) + \sum_{n=2}^{\infty}(n-1)|a_n|^2 + b_2^2(1 - \sum_{n=2}^{\infty}|a_n|)^2}}{1 - \sum_{n=2}^{\infty}|a_n|} < b_1 - \alpha,$$

which is equivalent to

$$\sqrt{\left((b_1 - 1) + \sum_{n=2}^{\infty}(n - b_1)|a_n|\right)^2 + b_2^2\left(1 - \sum_{n=2}^{\infty}|a_n|\right)^2}$$

$$< b_1 - \alpha - \sum_{n=2}^{\infty}(b_1 - \alpha)|a_n|.$$

By using the same manner as in the proof of the previous theorem, we have

$$\left(\sum_{n=2}^{\infty}(n-\alpha)|a_n|\right)\left(\sum_{n=2}^{\infty}(n - 2b_1 + \alpha)|a_n|\right) + b_2^2\left(\sum_{n=2}^{\infty}|a_n|\right)^2 + 2(b_1 - 1)\sum_{n=2}^{\infty} n|a_n|$$

$$+ (2b_1 - 4\alpha b_1 + 2\alpha^2 - 2b_2^2)\sum_{n=2}^{\infty}|a_n|$$

$$< 2b_1 - 2\alpha b_1 + \alpha^2 - b_2^2 - 1.$$

Therefore, we have

$$\left(\sum_{n=2}^{\infty}(n-\alpha)|a_n| + \frac{(\alpha-b_1)^2-|b|^2+b_1}{\alpha-b_1}\right)\left(\sum_{n=2}^{\infty}(n-2b_1+\alpha)|a_n| + \frac{(\alpha-b_1)^2-|b|^2+b_1}{\alpha-b_1}\right)$$

$$+b_2^2\left(\sum_{n=2}^{\infty}|a_n|\right)^2 + 2(b_1-1)\sum_{n=2}^{\infty}n|a_n|$$

$$< |b|^2 - 1 - \frac{|b|^2(|b|^2-2b_1)-b_1^2}{(\alpha-b_1)^2}.$$

This completes the proof of the theorem. □

V. Some examples of functions belonging to the class $S(\alpha, b)$.

Now, we consider some examples of functions $f(z)$ which belong to the class $S(\alpha, b)$.

Remark 5. Since

$$f(z) \in S(\alpha, b) \Longleftrightarrow \left|\frac{zf'(z)}{f(z)} - b\right| < \mathrm{Re}(b) - \alpha,$$

we consider B which satisfies

$$\left|\frac{zf'(z)}{f(z)} - b\right| < |1-b| + |B| = \mathrm{Re}(b) - \alpha.$$

Thus we get

$$|B| = \mathrm{Re}(b) - \alpha - |1-b|,$$

and

$$B = (\mathrm{Re}(b) - \alpha - |1-b|)\, e^{i\phi}.$$

For such B, we consider

$$\frac{zf'(z)}{f(z)} - b = (1-b) + Bz,$$

which gives that

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = B.$$

Integrating both sides, we have

$$\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right) dt = B\int_0^z dt,$$

so

$$\frac{f(z)}{z} = e^{Bz}.$$

Thus we obtain

$$f(z) = ze^{\{(\mathrm{Re}(b)-\alpha-|1-b|)e^{i\phi}\}z}.$$

Taking some b and α in Remark 5, we give

Example 3. *Taking* $b = 1 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, *we have*

$$f(z) = ze^{\{(\mathrm{Re}(b)-\alpha-|1-b|)e^{i\phi}\}z} = ze^{\frac{1}{6}z}.$$

Since

$$f'(z) = \left(1 + \frac{1}{6}z\right) e^{\frac{1}{6}z}$$

we see

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{6}z.$$

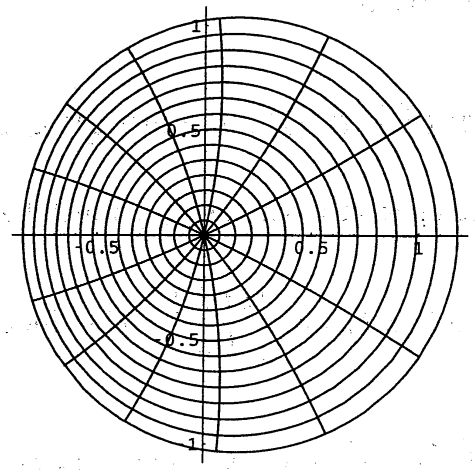

Figure 7: Image of U by $f(z)$ for Example 3.

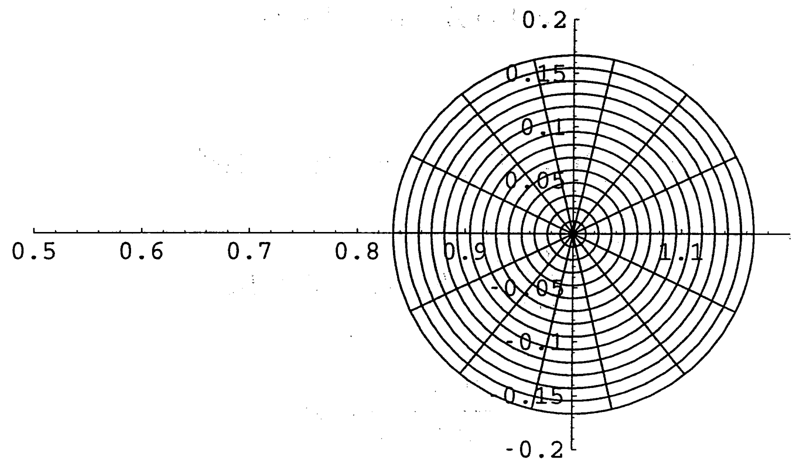

Figure 8: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 3.

Example 4. *Taking* $b = 2 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, *we have*

$$f(z) = ze^{\{(\mathrm{Re}(b)-\alpha-|1-b|)e^{i\phi}\}z} = ze^{\left(\frac{3}{2}-\frac{\sqrt{10}}{3}\right)z}.$$

Since

$$f'(z) = \frac{1}{6}(6 + (9 - 2\sqrt{10})z)e^{\left(\frac{3}{2} - \frac{\sqrt{10}}{3}\right)z}$$

we see

$$\frac{zf'(z)}{f(z)} = 1 + \left(\frac{3}{2} - \frac{\sqrt{10}}{3}\right) z.$$

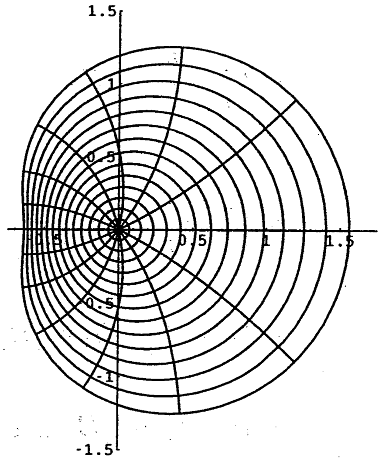

Figure 9: Image of U by $f(z)$ for Example 4.

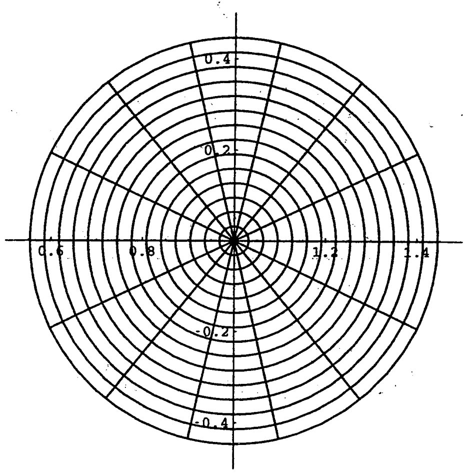

Figure 10: Image of U by $\dfrac{zf'(z)}{f(z)}$ for Example 4.

Next, we consider

Remark 6. We consider B such that

$$\frac{zf'(z)}{f(z)} - b = \frac{(1-b) + Bz}{1-z}.$$

Noting that

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-b\right| &= \left|\frac{1-b+Bz}{1-z}\right| \\ &= \left|\frac{1-b}{1-z}+\frac{Bz}{1-z}\right| \\ &= \left|\frac{1-b}{1-z}+\frac{B}{1-z}-B\right| \\ &< \left|\frac{1-b}{1-z}\right|+\left|\frac{B}{1-z}\right|+|B| \\ &< |1-b|+|B|+|B| \\ &= |1-b|+2|B| \leqq \mathrm{Re}(b)-\alpha,\end{aligned}$$

we find B given by

$$|B| = \frac{\mathrm{Re}(b)-\alpha-|1-b|}{2},$$

$$B \;= \left(\frac{\mathrm{Re}(b)-\alpha-|1-b|}{2}\right)e^{i\phi}.$$

For such B, we see that

$$\frac{f'(z)}{f(z)}-\frac{1}{z}=\frac{1-b+B}{1-z}.$$

After integration, we have

$$\log\frac{f(z)}{z}=\log\,(1-z)^{(b-1-B)},$$

that is

$$f(z)=z(1-z)^{(b-1-B)}.$$

It follows that

$$f(z)=z(1-z)^{\left(b-1-\left(\frac{\mathrm{Re}(b)-\alpha-|1-b|}{2}\right)e^{i\phi}\right)}.$$

Taking some b and α, Remark 6 gives us

Example 5. *Making* $b=1+\dfrac{1}{100}i$, $\alpha=\dfrac{1}{2}$, *we have*

$$f(z) = z(1-z)^{\left(b-1-\left(\frac{\mathrm{Re}(b)-\alpha-|1-b|}{2}\right)e^{i\phi}\right)} = z(1-z)^{\frac{1}{200}(2i-49(-1)^{\frac{1}{100}})},$$

$$f'(z)=\frac{1}{200}(200+((-200-2i)+49(-1)^{\frac{1}{100}})z)(1-z)^{\frac{1}{200}((-200+2i)-49(-1)^{\frac{1}{100}})}.$$

This gives us that

$$\frac{zf'(z)}{f(z)} = \frac{200 - ((200 + 2i) - 49(-1)^{\frac{1}{100}})z}{200(1-z)}.$$

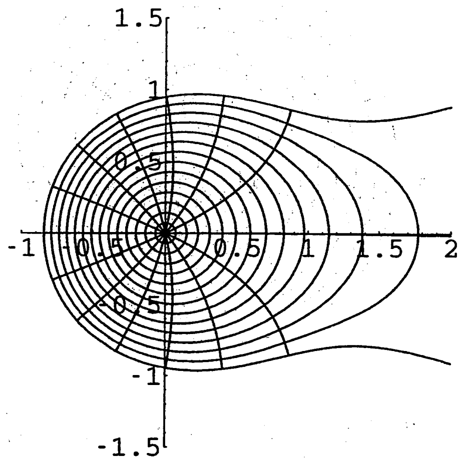

Figure 11: Image of U by $f(z)$ for Example 5.

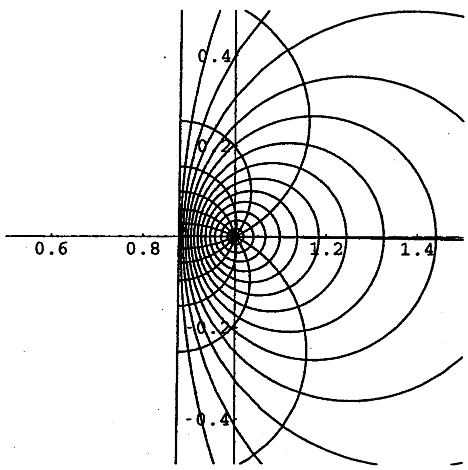

Figure 12: Image of U by $\dfrac{zf'(z)}{f(z)}$ for Example 5.

Example 6. *Making* $b = 2 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, *we have*

$$f(z) = z(1-z)^{\left(b-1-\left(\frac{\mathrm{Re}(b)-\alpha-|1-b|}{2}\right)e^{i\phi}\right)} = z(1-z)^{\frac{1}{12}(3+2\sqrt{10}+4\,i)},$$

$$f'(z) = -\frac{1}{12}(-12 + (15 + 2\sqrt{10} + 4\,i)z)(1-z)^{\frac{1}{12}(-9+2\sqrt{10}+4\,i)}.$$

This gives us that

$$\frac{zf'(z)}{f(z)} = \frac{12 - (15 + 2\sqrt{10} + 4\,i)z}{12(1-z)}.$$

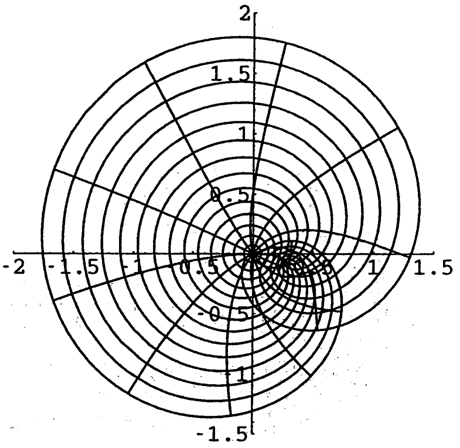

Figure 13: Image of U by $f(z)$ for Example 6.

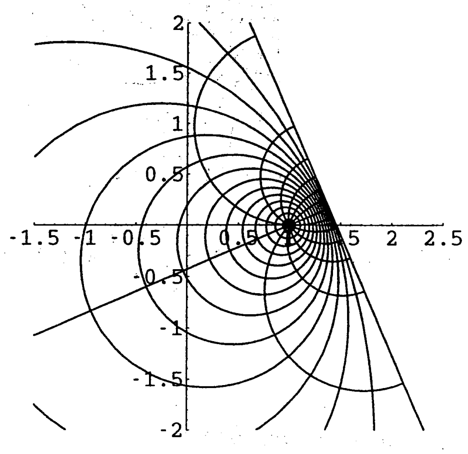

Figure 14: Image of U by $\dfrac{zf'(z)}{f(z)}$ for Example 6.

Finally, considering $f(z) = z + a_2 z^2$, we derive

Remark 7. Let us consider the function $f(z)$ given by $f(z) = z + a_2 z^2$. For such $f(z)$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1+2a_2 z}{1+a_2 z} = 1 + \frac{a_2 z}{1+a_2 z}$$

$$= 1 + \frac{B}{1+B},$$

where $B = a_2 z$. Since

$$\left|\frac{zf'(z)}{f(z)} - b\right| = \left|\frac{B}{1+B} + 1 - b\right| < \mathrm{Re}(b) - \alpha,$$

we have

$$\left|\frac{B}{1+B} + 1 - b\right|^2 < (\mathrm{Re}(b) - \alpha)^2.$$

Let $C = (\mathrm{Re}(b) - \alpha)^2$. A simple calculation gives that

$$|B|^2 + \frac{(1-b)(2-\bar{b}) - C}{|2-b|^2 - C}\bar{B} + \frac{(1-\bar{b})(2-b) - C}{|2-b|^2 - C}B < \frac{C - |1-b|^2}{|2-b|^2 - C},$$

so

$$\left|B + \frac{(1-b)(2-\bar{b}) - C}{|2-b|^2 - C}\right|^2 = \left|B - \frac{C - (1-b)(2-\bar{b})}{|2-b|^2 - C}\right|^2$$

$$< \frac{1}{(|2-b|^2 - C)^2}(|2-b|^2 C - |1-b|^2|2-b|^2 + |1-b|^2 C$$

$$+ |(1-b)(2-\bar{b})|^2 - (1-b)(2-\bar{b})C - (1-\bar{b})(2-b)C)$$

$$= \frac{C}{(|2-b|^2 - C)^2}.$$

Thus we see

$$\left|B - \frac{C - (1-b)(2-\bar{b})}{|2-b|^2 - C}\right| < \frac{\sqrt{C}}{|2-b|^2 - C}.$$

Consequently, we obtain

$$\left|B - \frac{(\mathrm{Re}(b) - \alpha)^2 - (1-b)(2-\bar{b})}{|2-b|^2 - (\mathrm{Re}(b) - \alpha)^2}\right| < \frac{\sqrt{(\mathrm{Re}(b) - \alpha)^2}}{|2-b|^2 - (\mathrm{Re}(b) - \alpha)^2}$$

$$= \frac{\mathrm{Re}(b) - \alpha}{|2-b|^2 - (\mathrm{Re}(b) - \alpha)^2}.$$

This shows that $B = a_2 z$ is inside of the cercle with the center at

$$\frac{(\mathrm{Re}(b) - \alpha)^2 - (1-b)(2-\bar{b})}{|2-b|^2 - (\mathrm{Re}(b) - \alpha)^2}$$

and radius

$$\frac{\mathrm{Re}(b) - \alpha}{|2 - b|^2 - (\mathrm{Re}(b) - \alpha)^2}.$$

Making use of $b = 1 + \frac{1}{3}i$ and $\alpha = \frac{1}{2}$ in Remark 7, we see

Example 7. *Taking $b = 1 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, we see that a_2 is inside of the circle with the center at $\frac{5}{31} + \frac{12}{31}i$, and radius $\frac{18}{31}$. Thus we have $-\frac{5}{31} = \frac{13}{31} - \frac{18}{31} < 0 < |a_2 z| < \frac{13}{31} + \frac{18}{31} = 1$. If we take $a_2 = \frac{1}{2}$, then*

$$f(z) = z + \frac{1}{2}z^2$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{z}{2+z}.$$

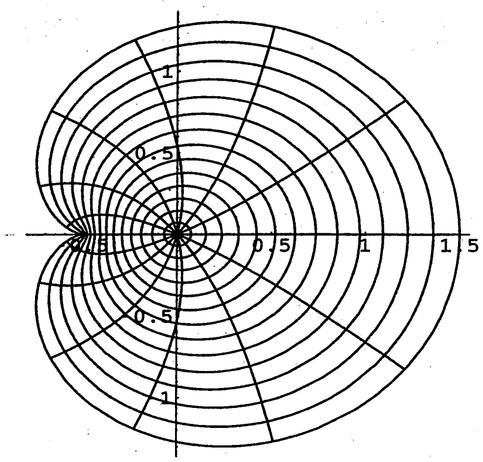

Figure 15: Image of U by $f(z)$ for Example 7.

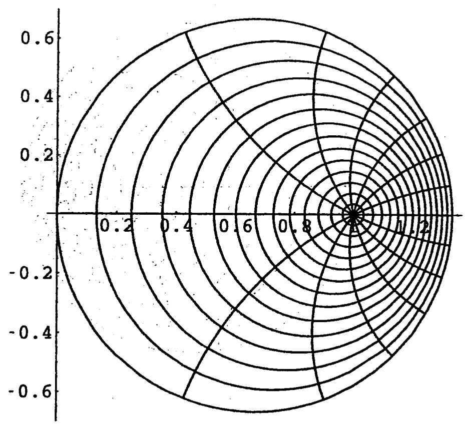

Figure 16: Image of U by $\dfrac{zf'(z)}{f(z)}$ for Example 7.

Also, letting $b = 2 + \dfrac{1}{3}i$ and $\alpha = \dfrac{1}{2}$, Remark 7 leads us

Example 8. *Taking* $b = 2 + \dfrac{1}{3}i$, $\alpha = \dfrac{1}{2}$, *we see that* a_2 *is inside of the circle with the center at* $1 + \dfrac{12}{77}i$, *and radius* $\dfrac{54}{77}$. *Thus we have* $0.310772 = \dfrac{\sqrt{6073}}{77} - \dfrac{54}{77} < |a_2 z| < 1 < \dfrac{\sqrt{6073}}{77} + \dfrac{54}{77} = 1.71337$. *If we take* $a_2 = \dfrac{2}{3}$, *then*

$$f(z) = z + \frac{2}{3}z^2$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2z}{3+2z}.$$

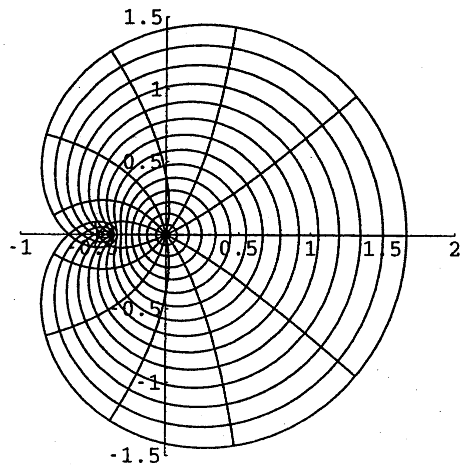

Figure 17: Image of U by $f(z)$ for Example 8.

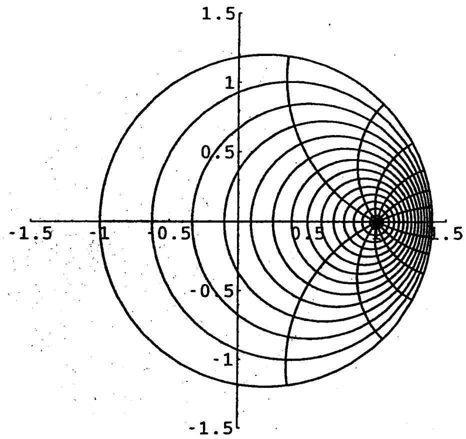

Figure 18: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 8.

References

[1] T. Sekine and S. Owa, Certain subclasses of starlike functions of order α, *PanAmer. Math. J.* **5**(1995), 95 - 100.

[2] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51**(1975), 109 - 116.

[3] H. Silverman, Starlike functions of positive order, *Bull. Inst. Math. Acad. Siniea* **28**(2000), 183 - 187.

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